FUSION PRODUCT OF CO-ADJOINT ORBITS

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1. Introduction

In this paper, we introduce the fusion product of a generic pair of co-adjoint orbits. This construction provides the geometric dual object to the product in Verlinde fusion algebra. The latter is a quantum deformation of the standard tensor product, the fusion product constructed here is the corresponding deformation of the Cartesian product.

Let G be a connected and simply-connected compact simple Lie group, P_+ be the set of dominant integral weights, θ be the highest root. Suppose $\lambda \in P_+$, and denote by $[\lambda]$ the irreducible G-module defined by λ. Let $(\cdot|\cdot)$ be the invariant bilinear form on $\mathfrak{g}^*, \mathfrak{g}$, normalized so that $(\theta|\theta) = 2$. For $k \in \mathbb{Z}_+$, define

$$P_+^k = \{\lambda \in P_+ |\, (\theta|\lambda) \leq k\}.$$

Denote by $\widetilde{LG}$ the central extension of the loop group LG.

The set P_+^k has 1-1 correspondence with the set of highest weight $\widetilde{LG}$-modules at level k. As a consequence of conformal field theory, one obtains a product on the set of integrable highest weight representations at each level. This product implies something new for the representations of G itself. Namely it induces the fusion product $[\lambda]\widehat{\otimes}_k[\lambda']$, for a pair $[\lambda], [\lambda']$ whenever $\lambda, \lambda' \in P_+^k$.

The fusion tensor has the property that all the dominant integral weights appearing in it are in the set P_+^k; and when the level is high enough, i.e., $k \geq (\lambda+\lambda'|\theta)$, the fusion tensor agrees with the standard one. It was conjectured by Verlinde the character of $[\lambda]\widehat{\otimes}_k[\lambda']$ satisfies the following:

$$\chi_{[\lambda]\widehat{\otimes}_k[\lambda']} = \chi_\lambda \cdot \chi_{\lambda'} \quad \text{on} \quad \exp(2\pi i \frac{M^*}{k+h^{\mathrm{v}}})$$

where M^* is the dual of long root lattice and h^{v} is the dual Coxeter number. Because the left side has only $\chi_c, c \in P_+^k$ in its expansion; and $\{\chi_c | c \in P_+^k\}$ is an orthonormal basis of functions on the set $\{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^{\mathrm{v}}}} | \lambda \in P_+^k\}$, with repsect to a suitable measure, the above uniquely determines the function $\chi_{[\lambda]\widehat{\otimes}_k[\lambda']}$.

This paper will give a proof of this identity. The more interesting result here is the concrete realization of $[\lambda]\widehat{\otimes}_k[\lambda']$ as holomorphic sections over the fusion product.

To understand the fusion product in constrast with the Cartesian product, we first recall the role played by the latter in the representations of G.

Let $M_\lambda, M_{\lambda'}$ be the co-adjoint orbits passing λ, λ'. Let L_λ be the line bundle over M_λ defined by the character λ. Correspondingly, there is $L_{\lambda'}$ over $M_{\lambda'}$.

It is well known, through Borel-Weil theory, $[\lambda]$ can be realized as $H^0(M_\lambda, L_\lambda)$. Obviously then $[\lambda] \otimes [\lambda']$ can be realized as $H^0(M_\lambda \times M_{\lambda'}, L_\lambda \otimes L_{\lambda'})$.

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Hence arises the question as how to describe the new represenation $[\lambda]\widehat{\otimes}_k[\lambda']$ in a geometric manner.

For a generic pair $\lambda, \lambda' \in P_+^k$, the fusion product Y is obtained by performing certain surgery on the product of conjugacy classes

$$\mathrm{Ad}_G\, e^{2\pi i \lambda/k} \times \mathrm{Ad}_G\, e^{2\pi i \lambda'/k}.$$

The fusion product is a holomorphic G-orbifold with a orbifold line bundle V, and has the following property:

$$\mathrm{tr}(s|H^0(Y,V^k)) = \mathrm{RR}(H^0(Y,V^k))(s) = \chi_\lambda(s)\cdot\chi_{\lambda'}(s) \quad \text{on} \quad \exp(2\pi i \frac{M^*}{k+h^{\mathrm{v}}}).$$

One essential ingredient in verifying the above results is the new fixed point formula for loop group action proved in [C2]. The explicit calculation of the weights at the fixed points of the fusion product, which uses properties of affine roots transformed by affine Weyl group, is presented here for the first time.

The first step is to construct moduli space $\mathcal{M}_{(a,b)}$ of flat connections over three-holed Riemann sphere, with fixed holonomies around two circles and no constraint on the third one. It is a LG-space, and its quotient by the nilpotent subgroup $LG^{\mathbb{C}+}$ has an T-orbifold as its compactification, following the main result in [C1]. The fusion product is defined as $Y = G\times_T X_N$, which can be described in terms of the product of conjugacy classes. Using the fixed point formula of [C2], we calculate the equivariant Riemann-Roch of the pair (Y, V^k) which agrees with what Verlinde conjectured.

There are much in common between Y and $\mathrm{Ad}_G\,\lambda\times\mathrm{Ad}_G\,\lambda'$. The interior fixed points of Y has 1-1 correspondence with the set of fixed points on the Cartesian product. Although the weights are different. It turns out the contributions of the corresponding fixed points to Riemann-Roch differ by a factor which is 1 on the set $\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^{\mathrm{v}}}}|\lambda\in P_+^k\}$, as asserted by Theorem 4.1.

Another important difference is that for the standard tensor, there exists one highest weight, which is not true for the fusion tensor. The figure 4.2 illustrates this point.

For a generic pair λ, λ', we remark when the level $k \geq (\lambda+\lambda'|\theta)$, one does not automatically get back the product of the co-adoint orbits, or its diffeomorphic iamge. Instead one gets the 'twin' of the Cartesian product. The concept 'twin pair' was introduced in [C2, Sect. 12]. For a general compact G-symplectic manifold, with the prequantum data (M, L, f), such that $f(M)$ is transversal to $\mathfrak{t}$, it has a 'twin' (M_G, L_G) which has identical Riemann-Roch as that of (M, L). In general M_G is only symplectic outside a set of real codimension 2.

This raises the question whether there is another construction of the fusion product so that when the level $k\geq(\lambda+\lambda'|\theta)$, one gets back the Cartesian product. Although we know a potential candidate for it, there is some technical difficulty in proving the necessary properties.

1.1. **Relation with others work.** There have been a long list of work related to fusion product and the moduli of flat connections, see [Be, MS, L, JK].

The present work has one advantage that we have found a a compact holomorphic model (X_N) which has all the relavant information about $\mathcal{M}_{(a,b)}$. And the fixed point formula proved in [C2] enables us to calculate explicitly and directly the Riemann-Roch of $Y = G\times_T X_N$. The variety X_N serves naturally as the compactified quotient of $\mathcal{M}_{(a,b)}$ by the nilpotent subgroup $LG^{\mathbb{C}+}$. In general quotient

of a variety by nilpotent group rarely exists, even in finite dimension. The existing work on moduli of parabolic bundles, and on extended moduli spaces of flat connections do not provide the fusion product or X_N. Hence the fusion tensor can not be realized geometrically as it is done here.

Recent work [L] is related in terms of calculating the Riemann-Roch number of moduli of parabolic bundles, at least for the $SU(n)$-case. The calculation of other invariants, e.g symplectic volumes, was worked out there for general G. In that regard, [JK] is also related.

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2. Moduli space of flat connections on a Riemann surface with boundary

Let S be a Riemann surface with boundary, the number of boundary components is n. Let G be a connected, simply-connected and compact simple Lie group. Let $\mathcal{A}, \mathcal{A}(\partial S)$ be the space of all smooth G-connections over $S, \partial S$ of a fixed G-principle bundle respectively, and $\mathcal{G}, \mathcal{G}(\partial S)$ be the group of smooth gauge transformations, $F_A = d_A A$ the curvature operator. Because S is of dimension 2, the principle bundle can be assumed to be the trivial one. Clearly $\mathcal{G}(\partial S) = LG^n$ where LG is the loop group.

2.1. **What topology?** Let $|\cdot|_{l,S}, |\cdot|_{k,\partial S}$ denote the norms of the Sobolev spaces $W^l(S), W^k(\partial S)$ over S and ∂S respectively.

Let $\mathcal{A}, \mathcal{G}$ be completed by $|\cdot|_{l-1,S} + |\cdot|_{k-1,\partial S}$ and $|\cdot|_{l,S} + |\cdot|_{k,\partial S}$ respectively, denote the completions by $\mathcal{A}^{l-1,k-1}(S), \mathcal{G}^{l,k}(S)$.

Then the gauge transformation is a bounded operator

$$\mathcal{A}^{l-1,k-1}(S) \times \mathcal{G}^{l,k}(S) \to \mathcal{A}^{l-1,k-1}(S).$$

Let $\mathcal{A}^{k-1}(\partial S), \mathcal{G}^k(\partial S)$ be the completions of $\mathcal{A}(\partial S), \mathcal{G}(\partial S)$ under $W^{k-1}(\partial S), W^k(\partial S)$ norms respectively.

We make the observation that the restriction

$$\begin{aligned} &A \in \mathcal{A}^{l-1,k-1}(S) \mapsto A|_{\partial S} \in \mathcal{A}^{k-1}(\partial S), \\ &g \in \mathcal{G}^{l,k}(S) \mapsto g|_{\partial S} \in \mathcal{G}^k(\partial S) \end{aligned} \tag{2.1}$$

are bounded operators. Each element $d + ad\theta \in \mathcal{A}^{k-1}(\partial S)$ has an extension to S inside $W^{k+1/2}(S)$, in particular it is in $\mathcal{A}^{l-1,k-1}(S)$ if $l \leq k + 1/2$. In fact the harmonic extension has this property by interior regularity result. In other words, the restriction map defined above is onto. Likewise for $\mathcal{G}^k(\partial S)$.

From now on, we assume that $k \geq 2, k+1/2 \geq l > 2$. This way, all the elements are continuous.

Applying the proof of Theorem 3.2 of [FU] to both $S, \partial S$, we conclude $\{g_i\}$ has subsequential convergence in $\mathcal{G}^{l,k}(S)$ if $g_i(A) \to B$ in $\mathcal{A}^{l-1,k-1}(S)$. Hence the orbit space is Hausdorff.

The tangent space of $\mathcal{A}^{l-1,k-1}(S)$ is given by

$$T_A\mathcal{A}^{l-1,k-1}(S) = \{a \in W^l(S, \mathfrak{g} \times T^*(S) \big| \, a|_{\partial S} \in \mathcal{A}^{k-1}(\partial S)\}. \tag{2.2}$$

Let $\mathcal{G}_I^{l,k}(S)$ be the subgroup of $\mathcal{G}^{l,k}(S)$ with I on the ∂S. The following can be verified in the same way as in the case $\partial S = \emptyset$, utilizing the condition that $\mathrm{Lie}\mathcal{G}_I^{l,k}(S)$ has 0 boundary values on ∂S:

Lemma 2.1. *The curvature operator $F : \mathcal{A}^{l-1,k-1}(S)\mathcal{A}^{l-1,k-1}$ with $F(A) = F_A = d_A A$ is a moment map for the action by $\mathcal{G}^{l,k}(S)$, with respect to the symplectic form Ω on $T\mathcal{A}^{l-1,k-1}(S)$ given by*

$$\Omega(a,b) = \frac{1}{2\pi}\int_S (a,b)$$

where $(,)$ is the same bilinear form on $\mathfrak{g}$ as before.

If $*$ is the Hodge operator defined by the Riemann surface, then $\Omega(\cdot, *\cdot)$ is positive definite.

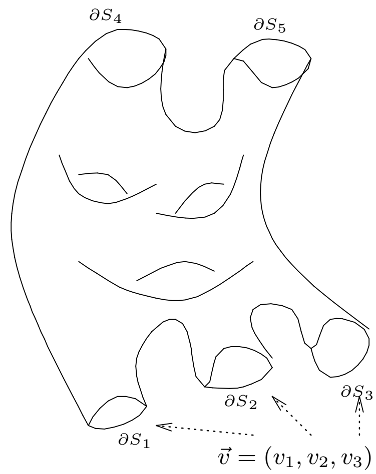

FIGURE 2.1. m=3, n=5

Proposition 2.1. *1). The space of flat connections mod out the action by $\mathcal{G}_I^{l,k}(S)$, $\mathcal{M} = F^{-1}(0)/\mathcal{G}_I^{l,k}(S)$, is an infinite dimensional complex Hilbert manifold. The tangent space at A is the space*

$$T_{[A]}\mathcal{M} = \{a \in T_A\mathcal{A}^{l-1,k-1}(S) |\, d_A a = d_A * a = 0\}.$$

2). The action by $(LG)^n \simeq \mathcal{G}^{l,k}(S)/\mathcal{G}_I^{l,k}(S) \simeq \mathcal{G}^k(\partial S)$ on $\mathcal{M}$, defined by extending each $g \in (LG)^n$ to an element in $\mathcal{G}^{l,k}(S)$, $\tilde{g}$, then $g[A] = [\tilde{g}A]$, is Hamiltonian with respect to the induced two form on $T_{[A]}\mathcal{M}$ from Ω. The moment map is given by $\Phi([A]) = a$ if $A_{\partial S} = d + ad\theta$ with $a \in (l\mathfrak{g})^n$.

3). There is a line bundle $\tilde{L}$ over $\mathcal{M}$ on which $\widetilde{LG}$ acts covering the action by LG on $\mathcal{M}$. It has a invariant connection whose curvature is $2\pi i\Omega$.

2.2. **Further symplectic reduction on $\mathcal{M}$.** Let $0 < m \leq n$, fix m components of ∂S, and the corresponding m-components Ψ in the moment map Φ. The action by different components of LG in $(LG)^n$ commutes, as it can be observed from the definition of its action on $\mathcal{M}$.

Clearly Ψ is the moment map associated with the action by $(LG)^m$ on the selected m-components in ∂S. Let $\vec{v} \in (l\mathfrak{g})^m$ be a fixed elements. And $(LG)^m_{\vec{v}}$ be the stabilizer of $d + \vec{v}d\theta$.

Proposition 2.2. *The space $\mathcal{M}_{\vec{v}} = \Psi^{-1}(\vec{v})/(LG)^m_{\vec{v}}$ is a complex Hilbert manifold on which $(LG)^{n-m}$ acts in a Hamiltonian manner. The line bundle $\tilde{L}$ descends to one on $\mathcal{M}_v$ denoted by L on which $(\widetilde{LG})^{n-m}$ acts. The two form Ω descends to ω on $\mathcal{M}_{\vec{v}}$. It is the $2\pi i$-curvature of an $(\widetilde{LG})^{n-m}$-invariant connection.*

Remark: One can combine the two above to get $\mathcal{M}_{\vec{v}}$ directly from $\mathcal{A}$. The action with respect to the full gauge transformation is Hamiltonian, the moment map is given by $(F_A, \Phi(A))$. Selecting m-boundary components. Then the action by the subgroup $\mathcal{G}^{l,k}(S)_m$ whose elements are I on the other $n-m$ components, is Hamiltonian with respect to the map (F, Ψ).

Given $\vec{v}$, define the subgroup $\mathcal{G}^{l,k}_{\vec{v}}(S) \subset \mathcal{G}^{l,k}(S)_m$ which consists of elements with boundary values in $(LG)^m_{\vec{v}}$ on the m-designated boundary components, and I on

the rest of boundary. The new subgroup is the stabilizer of $(0, \vec{v})$. The quotient $(F^{-1}(0), \Psi^{-1}(\vec{v}))$ by $\mathcal{G}^{l,k}_{\vec{v}}(S)$ is the desired space.

The proof of the above is a standard exercise.

Let's state an obvious fact:

Lemma 2.2. *The map $\mu : \mathcal{M}_{\vec{v}} \to (l\mathfrak{g})^{n-m}$, $\mu([A])$ given by the boundary values at the other $n-m$ components is proper. It is the moment map with respect to the action by LG^{n-m}.*

This is obtained by noticing that $\Psi^{-1}(C^m)$ is compact where C is the affine alcove of $\mathfrak{g}$.

2.3. **Transversality.**

Lemma 2.3. *Suppose $0 < m < n$, then for generic value $\vec{v} \in C^m$, $\mathcal{M}_{\vec{v}}$ satisfies the transversality condition.*

Pf: Write $\Psi = (\psi, \mu) \in l\mathfrak{g}^m \times l\mathfrak{g}^{n-m}$. First we show that the slice $\mu^{-1}(\mathfrak{t}^{n-m})$ is a smooth submanifold in $\mathcal{M}$. We verify the claim by showing if $\mu(A) \in (\mathfrak{t})^{n-m}$, $D\mu(T_A\mathcal{M})$ is onto $(l\mathfrak{g}/\mathfrak{t})^{n-m}$. Let $\mathfrak{k} = (l\mathfrak{g})^n_\Psi$, the stabilizer of $\Psi(A)$. It is a direct product of the stabilizer of each component in $\Psi(A)$. Hence it is finite dimensional. So $Z_\mathfrak{k} = \Psi^{-1}(\mathfrak{k})$ is finite dimensional, smooth and symplectic near A. And $d\Psi$ is onto $(l\mathfrak{g})^n/\mathfrak{k}$. Write $\mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2) \subset l\mathfrak{g}^m \times l\mathfrak{g}^{n-m}$. We need to check that $D_A\mu(T_A Z_\mathfrak{k})$ is onto $\mathfrak{k}_2/(\mathfrak{g})^{n-m}$. Otherwise, from finite dimensional theory, there is a $\eta \in \mathfrak{k}_2/(\mathfrak{t})^{n-m}$ fixing A, but if we take the extension of η as having 0 boundary value on the first m-components, we know that no flat connections having such a stabilizer. Thus $D_A\mu(T_A Z_\mathfrak{k})$ is onto $\mathfrak{k}_2/(\mathfrak{g})^{n-m}$. Or $D_A\mu(T_A\mathcal{M})$ is onto $l\mathfrak{g}/\mathfrak{t}$, and $\mu^{-1}(\mathfrak{t}^{n-m})$ is a smooth submanifold in $\mathcal{M}$. Apply Sard's theorem to

$$\psi : \mu^{-1}(\mathfrak{t}^{n-m}) \to l\mathfrak{g}^m,$$

thus for generic $\vec{v}$, $\psi^{-1}(\vec{v})$ is smooth, which also implies the action by $(LG)^m_{\vec{v}}$ on the first m-components has only discrete stabilizer. Therefore, we have the generic smoothness of $\mathcal{M}_{\vec{v}}$.

We may further assume that $\vec{v} \in C^m$, otherwise we may transform $\vec{v}$ there by conjugation, and the discussion above about the smoothness of $\psi^{-1}(\vec{v})$ remains valid under conjugation. QED

3. When S is a pair of pants

The situation when S is the sphere after removig three disjoint disks is of special interest.

Let $m = 2$, and $\vec{v} = (a, b)$ be a pair of elements in $\mathfrak{t} \times \mathfrak{t}$. From the last section, we know that for generic $\vec{v}$, $\mathcal{M}_{\vec{v}}$ is a comlex LG-manifold satisfying the transversality condition.

From [C1,C2] we have constructed a pair of T and G-orbifold $X_N, Y = G \times_T X_N$ respectively. So what are those spaces in the present situation?

Let $(x, y) = (\exp(2\pi ia), \exp(2\pi ib))$, $G_c^{\rm ss}$ denote the semi-simple subgroup commuting with $c \in T$. Naturaly $G_c^{\rm ss} \cap T$ is a maximal torus.

Proposition 3.1. *X_N is constructed from a subset of the Cartisian product of the conjugacy classes passing a, b respectively, then collapsing certain orbits of the subgroups in T:*

$$X_N = \{(r, s) \in \operatorname{Ad}_G x \times \operatorname{Ad}_G y |\, rs \in e^{2\pi iC}\}/ \simeq, \tag{3.1}$$

where $\simeq$ is defined by $(r, s) \simeq (tr, ts)$ for $t \in G_c^{\rm ss} \cap T$ where $c = rs$.

$Y = G \times_T X_N$ and it also can be described as a blow up of the following:

$$\{(r, s) \in \operatorname{Ad}_G x \times \operatorname{Ad}_G y\}/ \simeq$$

where $\simeq$ is define by $(r, s) \simeq (gr, gs)$ for $g \in G_c^{\rm ss} \cap \operatorname{Ad}_c T$ where $c = rs$.

With this description, one can see that Y is related but may not be diffeomorphic to $\operatorname{Ad}_G a \times \operatorname{Ad}_G b$.

Definition 3.1. *The fusion product of the co-adjoint orbits $\operatorname{Ad}_G a \times \operatorname{Ad}_G b$ is defined to be Y as in the above.*

3.1. Equivariant Riemann-Roch of Y.

Suppose $\lambda, \lambda' \in P_k^+$. Suppose the pair $(a, b) = (\lambda/k, \lambda'/k)$ satifies the generic condition, the moduli space $\mathcal{M}_{(a,b)}$ and the line bundle L^k induce the space Y and a holomorphic G-line bundle V over Y.

Let $\chi(g)$ be the character of the representation of G on $H^0(Y, V^k)$, the equivariant Riemann-Roch of the pair be $\operatorname{RR}(Y, V^k)$.

3.2. Fixed point sets on the fusion product.

Given a $\tau \in \{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} |\lambda \in P_+^k\}$, let us identify the fixed points on $\mathcal{M}_{(a,b)}$. Each $[A] \in \mathcal{M}_{(a,b)}$ represents a class of flat connection. Choose a base point p on the third boundary component and evaluate the holonomy of the connection A with that base point. One gets a pair $(x, y) \in \operatorname{Ad}_G e^{2\pi ia} \times \operatorname{Ad}_G e^{2\pi ib}$. Suppose the image $\mu(A)$ is in $\mathfrak{t}$. That A is fixed by τ implies (x, y) is fixed by τ under conjugation. Because τ is a regular element in T, we know that $x, y \in T$. Hence $(x, y) \in W(e^{2\pi ia}) \times W(e^{2\pi ib})$. Therefore all the flat connections fixed by τ produces such representations. In this case, the τ-fixed points and T-fixed points are the same.

Suppose $[A']$ has the same holonomy representation. Then there is a gauge transformation g on S such that $g(p) \in T$, and $g(A') = A$. Since $\mu([A]), \mu([A'])$ are fixed by τ, they are in $\mathfrak{t}$. Or

$$\mu([A]) = \mu(g[A']) = \operatorname{Ad}_g \mu([A']) \in \mathfrak{t}.$$

One concludes that

$$g^{-1}dg + g^{-1}\mu g \in \mathfrak{t}, \quad g(p) \in T.$$

Therefore $g^{-1}(p)$ is an element in the co-root lattice. Thus if we requie $\mu(A) \in W(C \setminus C^{\mathrm{aff}})$, then there is only one $[A]$ which gives the desired representation. The holonomy of A around the the third boundary component is given by z with

$$z = e^{2\pi i \mu(A)} = u(e^{2\pi i a}) \cdot v(e^{2\pi i b}). \tag{3.2}$$

One knows further that the image $\mu(A) \in W(C^{\mathrm{int}})$. Otherwise, $\mu(A)$ has non-trivial $(LG)^{\mathrm{ss}}_{\mu}$, and clearly $T \subset (LG)_{\mu}$. We know $T \subset (LG)_A$, because A is fixed by τ iff A is fixed by T. Therefore

$$[l\mathfrak{g}_{\mu}, l\mathfrak{g}_{\mu}] \cap l\mathfrak{g}_A \neq 0,$$

which violates the transversality condition.

Now we end the discussion by stating

Lemma 3.1. *The τ-fixed points on $\mathcal{M}_{(a,b)}$ are 1-1 correspondence with the set $W(e^{2\pi i a}) \times W(e^{2\pi i b})$. The image of the moment map of the element $[A]$ with $\mu(A) \in W(C^{\mathrm{int}})$ is the unique solution to Eq. 3.2*

Such fixed points on $\mathcal{M}_{(a,b)}$ induces the same on Y. In the current situation there is no need to consider the closure of the τ-fixed points on Y intersecting the compactfying locus, since all of them having images in $W(C^{\mathrm{int}})$. Therefore the remainder term $\mathcal{R}(\tau)$ defined in [C2] is 0 for all $\tau \in \{e^{2\pi i \nu^{-1} \frac{\lambda+\rho}{k+h^{\vee}}} | \lambda \in P^k_+\}$.

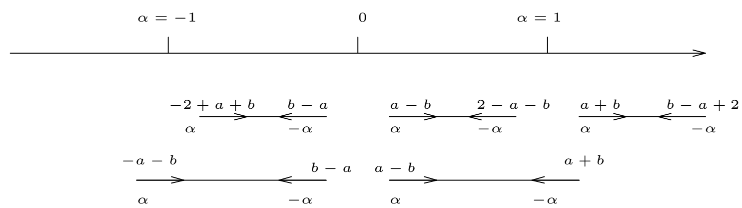

FIGURE 4.1. Weights at the fixed points of Cartesian and fusion products for $SU(2)$-case. The middle one is part of $\mu(\mathcal{M}_{(a,b)}) \cap \mathfrak{t}$ and the bottom one is $\phi(\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b) \cap \mathfrak{t}$.

4. WEIGHTS AT THE FIXED POINTS AND RIEMANN-ROCH

Compare the result on fixed points from the previous section, with the fixed points in the product of the coadjoint orbits, $\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b$, one sees the 1-1 correspondence between the fixed point sets of $\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b$ and the interior fixed points of the fusion product.

For the product of the coadjoint orbits, the image of a fixed point is $u(a) + v(b)$, while for the fusion product, the image is $c \in W(C^{\mathrm{int}})$ with $e^{2\pi ic} = u(e^{2\pi ia})\cdot v(e^{2\pi ib})$ or $c = u(a) + v(b) + t$ where t is a translation element in W^{aff}, with $e^{2\pi it} = I$.

In the following p denote a fixed point on the fusion product and q the corresponding fixed point on the Cartesian product.

Let $A \in \mathcal{M}_{(a,b)}$ be a fixed point, we want to calculate the weights on its tangent space. As the recipe provided in [C2] shows that the information on the weights and the fixed points will determine the equivariant Riemann-Roch of Y, hence the multiplicity of $\widetilde{LG}$, G-irreducible highest weight module in $H^0(\mathcal{M}_{(a,b)}, L^k), H^0(Y, V)$ respectively.

First let's examine the $G = SU(2)$ case.

Lemma 4.1. *1). For $G = SU(2)$, $C = [0, 1]$. For $a, b \in C$, the four fixed points on $\mathcal{M}_{(a,b)}$ with images in $[-1, 1] = W(C)$ are*

$$(e^{2\pi ai}, e^{2\pi bi}),\ (e^{-2\pi ai}, e^{2\pi bi}),\ (e^{2\pi ai}, e^{-2\pi bi}),\ (e^{-2\pi ai}, e^{-2\pi bi}).$$

2). The weight on T_AX_N which has $\dim_{\mathbb{C}} = 1$, is always given by $\pm\alpha$ where α is the root. Two out of four fixed points have images in C. Suppose $a \geq b$, then $c(e^{2\pi ai}, e^{-2\pi bi}) = a - b \in C$ the weight there is α.

$$\begin{cases} a+b<1: & c(e^{2\pi ai}, e^{2\pi bi}) = a+b \in C; \\ a+b>1: & c(e^{-2\pi ai}, e^{-2\pi bi}) = 2-(a+b) \in C, \end{cases} \tag{4.1}$$

the weights on T_AX_N in both cases are given by α.

Pf: Part 1 is obvious.

2). Fig. 4.1 illustrates what is going on here.

Remark: The weights in the figure have the opposite signs from what appear below in the text, because the weights discussed here refer to the weights by $t \in T$ on the tangent space, while what appear in the figure are those by the action of t^{-}.

The end points of C, $\alpha = 0, 1$ are identified with $0, \alpha^{\mathrm{v}}/2$. Since $e^0 = I, e^{\pi i\alpha^{\mathrm{v}}} = -I$ are in the center of the group, the image of $\mathcal{M}_{(a,b)}$ does not intersect ∂C. Otherwise we have rs is in the center of the group, and r, s after conjugating by

the same element in G, we may assume $r, s \in T$. Therefore A is fixed by T, and we have

$$(l\mathfrak{g})_A \cap [l\mathfrak{g}_\mu, l\mathfrak{g}_m u] \supset \mathfrak{t} \cap su(2) = \mathfrak{t}$$

which violates the transversality condition. Thus we have $\mu(A) \cap \partial C = \emptyset$.

Each end point of an interval in $\mu(A) \cap C^{\text{int}}$ must be the image of a fixed point, from a basic property of the moment map. Only two fixed points have their images in C^{int}, as we have identified them. Thus the image is a single interval. (In fact the image $\mu(X) \cap C$ is always a convex polytope, an observation I first made in loop group setting. But we can prove this without much work in the present situation.)

The local property near $\mu(A)$ determine the signs of the weights, which is a general fact from symplectic geometry. In fact, $\mu(A)$ is a left end point then the weight is $-\alpha$, and it is a right end point, then the weight is α.

In both case of part 2), $a+b > a-b$, and $2-(a+b) > a-b$. Therefore the weight is always α. QED

As shown by the above, the sign of the weight has something to do with the length of $a+b$, it is important.

Next we write the above result in a form more convient for later use.

Let $u, v \in W$, and t be a translation by an element in the lattice M generated by $W(\theta^{\text{v}})$.

Corollary 4.1. *Let $\phi : \mathrm{Ad}_G\, a + \mathrm{Ad}_G\, b \to \mathfrak{g}^*$ be the moment map of the Cartesian product, and $\phi(q) = u(a) + v(b)$. Suppose $c(p) = u(a) + v(b) + t \in C$, then the weight on $T_A X_N$ differs with that at $T_q\phi^{-1}(\mathfrak{t})$ iff $| < \alpha^{\text{v}}, u(a) + v(b) > | > 1$.*

Pf: By assumption, we have $| < \alpha^{\text{v}}, u(a) > |, | < \alpha^{\text{v}}, v(b) > | < 1$, and

$$0 << \alpha^{\text{v}}, c(p) >=< \alpha^{\text{v}}, u(a) + v(b) + t >< 1.$$

We have seen that if $< \alpha^{\text{v}}, u(a) >, < \alpha^{\text{v}}, v(b) >$ have different signs, then $| < \alpha^{\text{v}}, u(a) + v(b) > | < 1$, and the weight at p is $-\alpha$. On the other hand the weight of $T_q\phi^{-1}(\mathfrak{t})$ is given by $-\alpha$ as well, as it can be checked easily. Thus they have the same sign.

If $< \alpha^{\text{v}}, u(a) >, < \alpha^{\text{v}}, v(b) >$ have the same sign $+$, then from the prevous lemma, the sign of the weight of $T_p\mu^{-1}(\mathfrak{t})$ agree with that of $T_q\phi^{-1}(\mathfrak{t}) \cap (\mathrm{Ad}_G\, a + \mathrm{Ad}_G\, b)$.

If both $< \alpha^{\text{v}}, u(a) >, < \alpha^{\text{v}}, v(b) >$ have sign $-$, then $\alpha(t) = 2$, and

$$| < \alpha^{\text{v}}, u(a) > + < \alpha^{\text{v}}, v(b) > | > 1.$$

From the lemma, the weight on $T_p\mu^{-1}(\mathfrak{t})$ is α while at $T_q\phi^{-1}(\mathfrak{t})$ it is $-\alpha$.

Hence the only time the weights on $T_p\mu^{-1}(\mathfrak{t}), T_q\phi^{-1}(\mathfrak{t})$ are different, for $\mu(p) = u(a) + v(b) + t$ and $\phi(q) = u(a) + v(b)$ is when the the said condition is met. QED

From Fig. 4.1, one can see in the middle figure, the intervals $[-2+a+b, b-a], [a+b, 2+b-a]$ looks the same. This is because inside $\mathcal{M}_{(a,b)}$,

$$\mu^{-1}([-2+a+b, b-a]) = R_{\alpha^{\text{v}}}\mu^{-1}([a+b, 2+b-a]),$$

where $R_{\alpha^{\text{v}}}$ defined by the translation element $\alpha^{\text{v}} \in W^{\text{aff}}$ acts on $\mathcal{M}_{(a,b)}$. It preserves the complex structure and commutes with the T-action, therefore the two subvarieties have the same weights at the corresponding points. And one can furthre tell the difference between the Cartesian and fusion products at the point with image $a+b$.

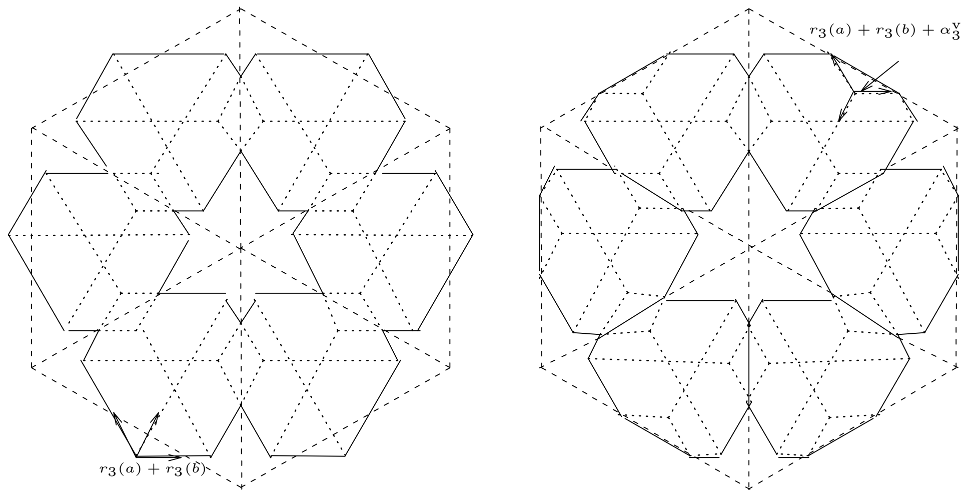

FIGURE 4.2. The left one is $\phi(\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b) \cap \mathfrak{t}$, the other is $\mu(Y) \cap \mathfrak{t}$.

4.1. **General G.** Now we want to compare the weights at for X_N, or $Y = G \times_T X_N$ with the Cartesian product. Fig. 4.1 illustrates the difference in $\mu(Y) \cap \mathfrak{t}$ and $\phi(\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b) \cap \mathfrak{t}$.

Here we further assume that $a, b \in C^{\mathrm{int}}$. Without that, e.g. b is a vertax $\neq 0$, then $e^{2\pi i b}$ is in the center of the group. Thus $\mathrm{Ad}_G\, e^{2\pi i b} = e^{2\pi i b}$, while $\mathrm{Ad}_G\, b$ has positive dimension. Therefore it is hard to compare the two varieties when one of a, b is on ∂C.

Obviously the condition is generic, and we will remove it in future.

The previous corollary can now be used to deal with the general cases.

From [K, Chap. 6], we know that M as a group of translation and the Weyl group of $\mathfrak{g}$, W, generate the affine Weyl group W^{aff}. And M is a normal subgroup. Thus each element of $\mathfrak{t}$ can be translated by an element in M to $W(C)$.

Suppose $(e^{2\pi u(a) i}, e^{2\pi v(b) i})$ defines a T-fixed point p with image in C, i.e. $\mu(p) = u(a) + v(b) + t \in C$. Then $\mu(p) \in C^{\mathrm{int}}$ by transversality. Let

$$q = (u(a), v(b)) \in \mathrm{Ad}_G\, u(a) \times \mathrm{Ad}_G\, v(b).$$

For each positive root α, the subalgebra $\mathfrak{g}_\alpha$ induces the subvariety $\mathrm{Ad}_{G_\alpha}\, e^{2\pi i u(a)} \times \mathrm{Ad}_{G_\alpha}\, e^{2\pi i v(b)}$ in $\mathcal{M}_{(a,b)}$, its intersection with $\mu^{-1}(\mathfrak{t})$ is symplectic at p, since $\mu(p) \in C^{\mathrm{int}}$.

For $a, b \in C^{\mathrm{int}}$, there is no semi-simple stabilizer of either $u(a), v(b)$. Thus the dimension of the subvariety is four and its intersection with $\mu^{-1}(\mathfrak{t})$ is of dimension 2 whose tangent space has a weight $\pm\alpha$.

The restriction of the two form Ω is positive definite to that tangent subspace, because the tangent vectors there are $\mathfrak{g}_\alpha$-valued 1-forms, and both T-action on the tangent space and the $*$-operator preserves the subspace at p.

The subvariety $\mathrm{Ad}_{G_\alpha}\, e^{2\pi i u(a)} \times \mathrm{Ad}_{G_\alpha}\, e^{2\pi i v(b)}$ is a G_α-variety. The restriction of the moment map μ to the subvariety the moment map for the G_α-action. Let $\mathfrak{t}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{t}$. Then for each pair $(r, s) \in \mathrm{Ad}_{G_\alpha}\, e^{2\pi i u(a)} \times \mathrm{Ad}_{G_\alpha}\, e^{2\pi i v(b)}$, if $e^{2\pi i \mu} = rs$,

μ is on the affine line

$$\mathfrak{t}_\alpha + \mu(p) = \mathfrak{t}_\alpha + (u(a) + v(b) + t).$$

Because the images pass $\mu(p)$ and must be in the direction of $\mathfrak{t}_\alpha$. As assumed $\mu(p) \in C$, thus $0 < (\alpha|\mu(p)) < 1$.

On the other hand, inside the Cartesian product of the coadjoint orbits, we have a similar subvariety $\mathrm{Ad}_{G_\alpha} u(a) \times \mathrm{Ad}_{G_\alpha} v(b)$. Its intersection with $\phi^{-1}(\mathfrak{t})$ is also symlectic, and with 2-d tangent space. The weight is given by $\pm\alpha$.

Proposition 4.1. *The weights on*

$$T_p(\mathrm{Ad}_{G_\alpha} e^{2\pi i u(a)} \times \mathrm{Ad}_{G_\alpha} e^{2\pi i v(b)}), \quad T_q(\mathrm{Ad}_{G_\alpha} u(a) \times \mathrm{Ad}_{G_\alpha} v(b))$$

differ iff

$$| < \alpha^{\mathrm{v}}, u(a) + v(b) > | > 1.$$

Pf: Let R_t denote the action by the translation element t in W^{aff}. As commented after Cor. 4.1, inside $\mathcal{M}_{(a,b)}$, the weights at T_p and $T_{R_{-t}p}$ are the same. Obviously $R_{-t}p$ is a T-fixed point.

The connection A which defines $R_{-t}p$, has boundary value at the 3rd component given by $\mu(A) = u(a) + v(b)$, since

$$\mu(R_{-t}p) = R_{-t}\mu(p) = R_{-t}(u(a) + v(b) + t) = u(a) + v(b)$$

by the equivariance of the moment map. Choose $a', b' \in \mathbb{R}_+ \simeq (\mathfrak{t}_\alpha)_+$, so that

$$< \alpha^{\mathrm{v}}, u(a) >= \pm a', \quad < \alpha^{\mathrm{v}}, v(b) >= \pm b'.$$

Now $< \alpha^{\mathrm{v}}, u(a) + v(b) >> 1$ iff a', b' have the same sign and $|a' + b'| > 1$. Then the variety $\mathrm{Ad}_{G_\alpha} e^{2\pi i a'} \times \mathrm{Ad}_{G_\alpha} e^{2\pi i b'}$ is exactly the same situation as the $SU(2)$-case, thus the weight has a different sign from that of T_q iff $|a' + b'| > 1$, or iff $< \alpha^{\mathrm{v}}, u(a) + v(b) >> 1$. QED

4.2. **Comparision of $\mathrm{FC}_p, \mathrm{FC}_q$ for general G.** Next we find another way of writing the condition when the sign of the weights differ.

All the facts about affine Lie algebra $\mathfrak{g}^{\mathrm{aff}}$ based on $\mathfrak{g}$ can be found in [K, Chap. 6]. The Lie algebra $\mathfrak{g}^{\mathrm{aff}}$ is the extention of $\widetilde{l\mathfrak{g}}$ by differentiation, d on the circle. The Lie algebra $\mathfrak{g}^{\mathrm{aff}}$ has $\mathfrak{t} \oplus \mathbb{R}K \oplus \mathbb{R}d$ as its Cartan subalgebra. Here K is the central element and d is the differentiation. The dual is given by $\mathfrak{t}^* \oplus \mathbb{R}\Lambda_0 \oplus \mathbb{R}\delta$. All the real positive affine roots are of the form

$$\Delta_+(\mathfrak{g}^{\mathrm{aff}}) = \Delta_+(\mathfrak{g}) \cup \{n\delta \pm \alpha | n \in \mathbb{Z}_+, \alpha \in \Delta_+(\mathfrak{g})\}.$$

It turns out the condition for when the signs of the weights differ is best written in the language of affine Lie algebras.

Assume $c = u(a) + v(b) + t \in C$, where t is in the coroot lattice, hence $0 < \alpha(c) < 1$, for all positive roots. As discussed earlier, the signs differ iff $|\alpha(u(a) + v(b))| > 1$.

1). $\alpha(c - t) = \alpha(u(a) + v(b)) < -1$. It can be written as $(\delta + \alpha)(R_{-t}c) < 0$. Thus the sign of α as a weight changes iff $(\delta + \alpha)(R_{-t}c) < 0$ or equivalently $R_t(\delta + \alpha) < 0$. In this case, at p the weight is α, at q it is $-\alpha$.

2). $\alpha(c - t) = \alpha(u(a) + v(b)) > 1$. It can be written as $(\delta - \alpha)(R_{-t}c) < 0$. In this case, the sign changes iff $R_t(\delta - \alpha) < 0$. And the weight is $-\alpha$ at p and α at q.

Are there other affine roots changing sign under R_t? The anwer is no. Because when $n \geq 2$,

$$n \pm < \alpha^{\mathrm{v}}, u(a) + v(b) > \geq n - 2 \geq 0.$$

Therefore the sign does not change under R_t.

Thus we obtain the following:

Lemma 4.2. *1). The sign changes iff $R_t(\gamma) < 0$ for a positive affine root $\gamma = \delta \pm \alpha$.*

2). Let the set of those roots whose induced weights on the tangent spaces at p, q differ by sign be denoted by S. Then $\sum_S \alpha = \sum_S \gamma \mod \delta = \sum_{R_t\gamma<0} \gamma \mod \delta$.

Next we compare $\det_{T_pY}(1-t^{-1})$ with $\det_{T_q(\mathrm{Ad}_G a\times \mathrm{Ad}_G b)}(1-t^{-1})$.

For $c \in C$, we have

$$\begin{aligned} &\det_{T_pY}(1-t^{-1}) = \prod_{\alpha\in\Delta_+(\mathfrak{g})}(1-e^{-\alpha}) \det_{T_p\mu^{-1}(\mathfrak{t})}(1-t^{-1});\\ &\det_{T_q(\mathrm{Ad}_G a\times \mathrm{Ad}_G b)}(1-t^{-1}) = \prod_{\alpha\in\Delta_+(\mathfrak{g})}(1-e^{-w\alpha}) \det_{T_q\phi^{-1}(\mathfrak{t})}(1-t^{-1}), \end{aligned} \tag{4.2}$$

where $w \in W$ in the above satisfies $\phi(q) = u(a) + v(b) = R_t(c) \in w(\mathfrak{t}_+)$.

Comparing the two in Eq. (4.2), we obtain for $s = e^{2\pi i\tau}$,

$$\begin{aligned} &\det_{T_q(\mathrm{Ad}_G a\times \mathrm{Ad}_G b)}(1-s^{-1})\\ &= (-1)^{|w|} e^{-\sum_{w\alpha<0} w\alpha} \prod_{\alpha\in\Delta_+(\mathfrak{g})}(1-e^{-\alpha}) \det_{T_q\phi^{-1}(\mathfrak{t})}(1-s^{-1})\\ &= (-1)^{|w|+\#S} e^{-\sum_{w\alpha<0} w\alpha} \prod_{\alpha\in\Delta_+(\mathfrak{g})}(1-e^{-\alpha}) e^{\sum_{\alpha\in S}\alpha} \det_{T_p\mu^{-1}(\mathfrak{t})}(1-s^{-1})\\ &= (-1)^{|w|+\#S} e^{-\sum_{w\alpha<0} w\alpha + \sum_{\alpha\in S}\alpha} \det_{T_pY}(1-s^{-1}). \end{aligned} \tag{4.3}$$

We already know that

$$\begin{aligned} &\sum_{\alpha\in S}\alpha = \sum_{R_t(\gamma)<0}\gamma \mod \delta;\\ &-\sum_{w\alpha<0, \alpha\in\Delta_+(\mathfrak{g})} w\alpha = \sum_{w^{-1}\beta<0, \beta\in\Delta_+(\mathfrak{g})}\beta. \end{aligned} \tag{4.4}$$

Now $w^{-1}\beta < 0$ iff $R_t(\beta) < 0$. To see that, write $\beta = -w\alpha$, then $w^{-1}\beta < 0$ iff $\alpha > 0$; we also know that

$$(R_t(\beta)|c) = (\beta|c-t) = -(w\alpha|c-t)$$

where $c - t \in w(\mathfrak{t}_+^*)$. Hence $R_t(\beta) < 0$ iff $-(w\alpha|c-t) < 0$, or iff $-(\alpha|\mathfrak{t}_+^*) < 0$. Thus

$$-\sum_{w\alpha<0, \alpha\in\Delta_+(\mathfrak{g})} w\alpha = \sum_{R_t\beta<0, \beta\in\Delta_+(\mathfrak{g})}\beta.$$

Furthermore, we have checked that no other positve affine roots change sign under R_t. Apply the well known formula:

$$(\rho + h^{\mathrm{v}}\Lambda_0) - w^{-1}(\rho + h^{\mathrm{v}}\Lambda_0) = \sum_{\gamma\in\Delta(\mathfrak{g}^{\mathrm{aff}}), w(\gamma)<0}\gamma, \quad \forall w \in W^{\mathrm{aff}}$$

we obtain the following key identity:

Proposition 4.2.

$$
\begin{aligned}
-\sum_{w\alpha<0} w\alpha + \sum_{\alpha\in S}\alpha &= \sum_{\beta\in\Delta_+(\mathfrak{g}^{\mathrm{aff}}),R_t\beta<0}\beta \mod \delta\\
&= -R_t^{-1}(\rho+h^{\mathrm{v}}\Lambda_0)+(\rho+h^{\mathrm{v}}\Lambda_0) \mod \delta\\
&= h^{\mathrm{v}}t \mod \delta;\\
(-1)^{|w|+\#S} &= (-1)^{|R_t|}\\
&= 1,
\end{aligned}
\tag{4.5}
$$

where the absolute value sign denotes the length of an element in W^{aff}. (Translation elements in W^{aff} have even lengths).

Now it is easy to prove the main result:

Theorem 4.1. *Let $\lambda,\lambda'\in P_+^k$, hence $a=\lambda/k, b=\lambda'/k\in C$.*

1). Let p,q denote the fixed points on Y, $\mathrm{Ad}_G\lambda\times\mathrm{Ad}_G\lambda'$ respectively, such that $c=\mu(p)=u(a)+v(b)+t\in C$, and $\phi(q)=u(a)+v(b)$. Let V be the same line bundle as before on Y, and the $L_\lambda\otimes L_{\lambda'}$ on $\mathrm{Ad}_G a\times\mathrm{Ad}_G b$. Denote by $\mathrm{FC}_p,\mathrm{FC}_q$ respectively the contributions of p,q to $\mathrm{RR}(Y,V^k),\mathrm{RR}(\mathrm{Ad}_G\lambda\times\mathrm{Ad}_G\lambda', L_\lambda\otimes L_{\lambda'})$. Then we have

$$FC_p(e^{2\pi i\tau}) = e^{2\pi i<(k+h^{\mathrm{v}})t,\tau>}\mathrm{FC}_q(e^{2\pi i\tau}).$$

2).

$$\mathrm{RR}(Y,V^k)(s)=\chi_\lambda(s)\cdot\chi_{\lambda'}(s) \quad \text{on} \quad \exp(2\pi i\frac{M^*}{k+h^{\mathrm{v}}}).$$

Remark: This proves Verlinde conjecture.

3). Write $\mathrm{RR}(Y,V)(s)=\sum_{l\in P_+^k} m_l s^l$, then m_c is given by

$$\frac{(-1)^l}{|\frac{M^*}{(k+h^{\mathrm{v}})M}|}\sum_{s\in\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^{\mathrm{v}}}}|\lambda\in P_+^k\}}\chi_{\bar c}(s)D^2(s)\chi_\lambda(s)\chi_{\lambda'}(s).$$

Pf: 1). The two functions $\mathrm{FC}_p,\mathrm{FC}_q$ can be written down as

$$
\begin{aligned}
\mathrm{FC}_p(e^{2\pi i\tau}) &= \frac{e^{2\pi ik<u(a)+v(b)+t,\tau>}}{\det_{T_pY}(1-e^{-2\pi i\tau})};\\
\mathrm{FC}_q(e^{2\pi i\tau}) &= \frac{e^{2\pi ik<u(a)+v(b),\tau>}}{\det_{T_q(\mathrm{Ad}_G a\times\mathrm{Ad}_G b)}(1-e^{-2\pi i\tau})}
\end{aligned}
\tag{4.6}
$$

we already have compared the denominators in the above, using Eq. (4.5) to obtain

$$
\begin{aligned}
\mathrm{FC}_p &= e^{2\pi ih^{\mathrm{v}}<t,\tau>}\frac{e^{2\pi ik<u(a)+v(b)+t,\tau>}}{\det_{T_q(\mathrm{Ad}_G a\times\mathrm{Ad}_G b)}(1-t^{-2\pi i\tau})}\\
&= e^{2\pi i<(h^{\mathrm{v}}+k)t,\tau>}\mathrm{FC}_q,
\end{aligned}
\tag{4.7}
$$

the two sides are equal when $<(h^{\mathrm{v}}+k)t,\tau>\in\mathbb{Z}$, or when

$$t\in\frac{M^*}{k+h^{\mathrm{v}}}.$$

Thus we have the conclusion.

2). Using fixed points formula proved in [C2], we obtain

$$\mathrm{RR}(Y, V)(e^{2\pi i\tau}) = \sum_p \mathrm{FC}_p(e^{2\pi i\tau}) \quad \text{on} \quad \frac{M^*}{k+h^\vee}, \tag{4.8}$$

which is amazing considering that Y has lots more fixed points than those interior ones. See figure on the right in fig. 4.2. All the fixed points with images on the boundary of wC of which there are plenty, do not appear in the above. Thus we have by Part 1),

$$\begin{aligned}
\mathrm{RR}(Y, V)(e^{2\pi i\tau}) &= \sum_p \mathrm{FC}_p(e^{2\pi i\tau}) \quad \text{on} \quad \frac{M^*}{k+h^\vee}, \\
&= \sum_q \mathrm{FC}_q(e^{2\pi i\tau}) \quad \text{on} \quad \frac{M^*}{k+h^\vee} \\
&= \mathrm{RR}(\mathrm{Ad}_G\, a \times \mathrm{Ad}_G\, b, L_a^k \otimes L_b^k) \quad \text{on} \quad \frac{M^*}{k+h^\vee} \\
&= \mathrm{RR}(\mathrm{Ad}_G\, \lambda \times \mathrm{Ad}_G\, \lambda', L_\lambda \otimes L_{\lambda'}) \quad \text{on} \quad \frac{M^*}{k+h^\vee} \\
&= \chi_\lambda \cdot \chi_{\lambda'} \quad \text{on} \quad \frac{M^*}{k+h^\vee}.
\end{aligned} \tag{4.9}$$

3). The expression follows right away from the expression of $\mathrm{RR}(Y, V^k)$ and Cor. 1.1 of [C2]. QED

5. The induced representation on the fusion product and the proof of an analogue of a conjecture by G. Segal

using the previous calculation of $\mathrm{RR}(Y, V^k)$. In order to do so, we need to know something about the higher cohomology groups. Ideally we want to show $H^i(Y, V^k) = 0, i > 0$. Or even better, the dual of the canonical line bundle $K^*(Y)$ is positive. That turns out to be more involved. Instead, we will consider the reduced space of Y, or of X_N, and use known results on the moduli of parabolic bundles.

5.1. Comparison of complex structures.

Recall $\mu : X_N \to kC \subset \mathfrak{t}$ is a moment map with respect to a two form with degeneracy along $k\partial C$. For $a \in kC$, let $X_c = \mu^{-1}(c)/T$, and L_c be the corresponding line bundle over it. Since X_N is an orbifold, for generic value of c, X_c is an orbifold. From the characterization of X_N in terms of the representation variety, it is easy to see that $X_c, c \in kC^{\mathrm{int}}$ is the same as that of representation variety

$$\{(h,k) \in \mathrm{Ad}_G\, e^{2\pi i a} \times \mathrm{Ad}_G\, e^{2\pi i b} | hk = e^{2\pi i c/k}\}/T.$$

That variety has a complex structure via its well known diffeomorphism with the moduli of parabolic bundles.

Proposition 5.1. *The two complex structure agree on X_c.*

Pf: First we describe the complex structure on X_c inherited from X_N. Below we shall use $D\mu, D_\phi$ to denote the differential of μ, ϕ so as not to confuse with the tangent vectors which are 1-forms.

What is $D_{(x,y)}(\mu - \mu_{\mathbb{X}})$? Recall that $T_q\mathbb{X} = l\mathfrak{g}/\mathfrak{t} \oplus T_z X_{\mathfrak{g}}$ from [C1], where $q = ([I, z])$. Let $\phi : X_{\mathfrak{g}} \to C \subset \mathfrak{t}$ as in [C1], be the moment map of the toric variety $X_{\mathfrak{g}}$.

The tangent space to $T_{[p,q]}X_c$ is given by

$$T_{[p,q]}X_c = \{(x,y) \in T_p\mathcal{M}_{(a,b)} \times l\mathfrak{g}/\mathfrak{t} | D_x\mu = D_y\mu_{\mathbb{X}}, D_{*x}\mu = D_{Jy}\mu_{\mathbb{X}}, D_y\phi = D_{Jy}\phi = 0\}.$$

As done in [C1], the complex structure J on $l\mathfrak{g}/\mathfrak{t}$ is given by $Jy = -y^J$ where y^J is the standard complex structure on $l\mathfrak{g}/\mathfrak{t}$, i.e. $y^J + iy$ is the boundary value of a holomorphic function on the unit disk.

We expand $y \in l\mathfrak{g}$ in terms of $E_\gamma, E_{-\gamma}$. The positive affine roots $\gamma \in \Delta_+(\mathfrak{g}^{\mathrm{aff}})$ are

$$\{n\delta \pm \alpha | n \geq 1, \alpha \in \Delta_+(\mathfrak{g})\} \cup \Delta_+(\mathfrak{g}^{\mathrm{aff}}) \cup \{n\delta | n \geq 1\}$$

and accordingly $E_\gamma = z^n E' \pm \alpha, E'_\alpha$ or $z^n h_\alpha$, where E'_α, h'_α are Chevalley basis of $\mathfrak{g}$.

The real form defining $l\mathfrak{g}$ has basis given by

$$x_\gamma = i(E_\gamma + E_{-\gamma}), \quad y_\gamma = E_\gamma - E_{-\gamma}.$$

By insisting on $y^J + iy$ being holomorphic, i.e. in $\sum_{\gamma>0} \mathbb{C}E_\gamma$, we obtain

$$x_\gamma^J = -y_\gamma, \quad y_\gamma^J = x_\gamma.$$

Let $y = \sum a_\gamma x_\gamma + b_\gamma y_\gamma$, then

$$\begin{aligned} y^J &= \sum -a_\gamma y_\gamma + b_\gamma x_\gamma \\ &= -\sum (a_\gamma - ib_\gamma)E_\gamma - (a_\gamma + ib_\gamma)E_{-\gamma}, \end{aligned} \tag{5.1}$$

and

$$D_{Jy}\mu = -D_{y^J}\mu = -\sum \gamma(\mu)(a_\gamma - ib_\gamma)E_\gamma + (a_\gamma + ib_\gamma)\gamma(\mu)E_{-\gamma}$$
$$= \sum a_\gamma |\gamma(\mu)| x_\gamma + b_\gamma |\gamma(\mu)| y_\gamma \tag{5.2}$$

where the fact $\gamma(\mu)$ is purely imaginary is used.

On the other hand, the complex structure derived from the parabolic bundles can be described by the following. Let $S^1 \times [0,\infty)$ be the conformal equivalent of the disk, with coordinate (θ, u), and $*du = -d\theta, *d\theta = du$. A 1-form a is in the tangent space to the parabolic bundle iff $d_A a = d_A * a = 0$ on the extended surface $S \cup S^1 \times [0,\infty)$, and a is in L^2.

Given x, y as in the above, we first extend y as a harmonic section with respect to $d_A * d_A$ on the cynlinder. Let $y = \sum a_\gamma x_\gamma + b_\gamma y_\gamma$, and define

$$\tilde{y} = \sum e^{-|\gamma(u)|}(a_\gamma x_\gamma + b_\gamma y_\gamma),$$

it is easy to check from this expression the extension is both harmonic (w.r.t $d_A * d_A$) and in L^2. Now the coefficients of $d\theta$ in $d_A \tilde{y}|_{\partial S}$ agrees with $d_x \mu$ by assumption. By direct calculation, the coefficent of du in $d_A \tilde{y}$ is given by

$$\sum a_\gamma |\gamma(\mu)| x_\gamma + b_\gamma |\gamma(\mu)| y_\gamma$$

which agree with the expression in Eq. 5.2, hence $x, d_A \tilde{y}$ agree on ∂S in both $d\theta, du$ directions. Therefore they define a L^2-harmonic 1-form with respect to d_A on the extended surface. From this, we conclude that the two complex structures agree. QED

5.2. **Vanishing result.** The positivity of $K^*(X_c)$ implies $H^i(X_c, L_c) = 0, i > 0$ for any semi-positive line bundle over X_c. We want to show

$$\operatorname{tr}(s|H^0(X_N, L_N)) = \mathrm{RR}(X_N, L_N) \tag{5.3}$$

which holds if $H^i(X_N, L_N) = 0, i > 0$ or having a positive $K^*(X_N)$. The proof of the positivity of $K^*(X_N)$ turns out to be more involved. So we shall by pass that to show Eq. 5.3.

We have $\dim H^0(X_c, L_c) = \mathrm{RR}(X_c, L_c)$ which equals the coefficient m_c of the character of weight c in $\mathrm{RR}(X_N, L_N)$, by the Abelian version of the result in [M]. We also know for X_N, that $\dim H^0(X_c, L_c)$ equals the coefficient of the character of weight c in $\operatorname{tr}(s|H^0(X_N, L_N))$, as shown in [GS]. Therefore,

$$\operatorname{tr}(s|H^0(X_N, L_N)) = \sum m_c s^c.$$

Apply the holomorphic induction to $Y = X \times_T X_N$, we know that

$$\operatorname{tr}(s|H^0(Y, L)) = \sum_W w \frac{\operatorname{tr}(s|H^0(X_N, L_N))}{\prod_{\alpha \in \Delta_+(\mathfrak{g})}(1 - s^{-\alpha})}$$
$$= \sum_c m_c \chi_c. \tag{5.4}$$

Now the Cor. 1.1 of [C2] gives an eplicit formula for m_c in terms of $\mathrm{RR}(Y, L)$:

$$
\begin{aligned}
m_c &= \frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{s \in \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}} \chi_{\bar{c}}(s) D^2(s) \mathrm{RR}(Y, L)(s) \\
&= \frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{s \in \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}} \chi_{\bar{c}}(s) \chi_\lambda(s) \cdot \chi_{\lambda'}(s) D^2(s).
\end{aligned} \tag{5.5}
$$

Thus the summation over $c \in P_+^k$ yields

(5.6)

$$
\begin{aligned}
\mathrm{tr}(s|H^0(Y, L)) &= \sum_c m_{c \in P_+^k} \chi_c(s) \\
&= \frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{c \in P_+^k, s \in \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}} \chi_c(s) \chi_{\bar{c}}(s) \chi_\lambda(s) \cdot \chi_{\lambda'}(s) D^2(s) \\
&= \chi_\lambda(s) \cdot \chi_{\lambda'}(s) \quad \text{on} \quad \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\},
\end{aligned}
$$

where we have used the following fact from [K, Chap. 13]:

$$
\frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{c \in P_+^k} m_c \chi_c(s) \chi_{\bar{c}}(s) D^2(s) = 1.
$$

Therefore we have verified the following conjectured by Verlinde in [V]:

Proposition 5.2.

$$
\chi_{[\lambda]\widehat{\otimes}_k[\lambda']}(s) = \mathrm{tr}(s|H^0(Y, L)) = \chi_\lambda(s) \cdot \chi_{\lambda'}(s) \quad \text{on} \quad \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}.
$$

For such λ, λ', after applying the result from [C1] and Eq. 5.4, we also obtain

Corollary 5.1.

$$
H^0(\mathcal{M}_{(a,b)}, L^k) = \sum_c m_c \tilde{\chi}_{(c,k)}
$$

which is analogous to a conjecture by G. Segal, see [T]. The original conjecture is made for certain moduli of holomorphic bundles instead of moduli of flat connections. Their equivalence can be established using argumenent similar to that of Donaldson for closed surface.

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